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Rotational graviton modes in the brane world

Justin F. Vázquez-Poritz*

Physique Théorique et Mathématique, Université Libre de Bruxelles
Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium
E-mail: poritz@student.physics.upenn.edu

Abstract: For a brane world embedded in various ten or eleven-dimensional geometries, we calculate the corrections to the four-dimensional gravitational potential due to graviton modes propagating in the extra dimensions, including those rotating around compact directions. Due to additional "warp" factors, these rotation modes may have as significant an effect as the s-wave modes which propagate in the large or infinite extra dimension.

Keywords: p-branes, Classical Theories of Gravity.

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1. Introduction

It has been proposed that there may exist extra dimensions which are almost the size of a millimeter [1, 2]. In this case, the gravitational potential would go as $1/r^{1+n}$ for sub-millimeter measurements, where n is the number of extra millimeter dimensions. The four-dimensional potential is recovered at larger scales. If the four-dimensional metric is multiplied by a "warp" factor which is a rapidly changing function of an extra dimension, then four-dimensional gravity is recovered at low-energy scales even for an infinite extra dimension [3, 4]. This is due to a massless graviton state that is bound to a 3-brane embedded in the higher-dimensional space. Large-energy modifications to the four-dimensional potential would result from the Kaluza-Klein modes propagating in the extra dimension.

It is possible that nature has chosen some extra dimensions to be infinite and others to be compact, especially if our spacetime is embedded within M-theory. For example, five-dimensional domain walls which localize gravity may arise from a sphere reduction from ten or eleven dimensions, as the near-horizon of extremal p-brane configurations [5].[1] In particular, it was found that such domain walls can be derived from a D3, D4 (or M5) and D5-brane, as well as two M5-branes intersecting over a 3-brane.[2]

In embedding five-dimensional brane world scenarios in higher dimensions, one generally only considers the s-wave Kaluza-Klein graviton mode. That is, as part of the dimensional reduction ansatz, one discards modes moving about the compact directions. It would appear to be a reasonable assumption that, unless the compact dimensions are quite large, the effects of such rotation modes on the corrections to the effective four-dimensional gravitation would be over-shadowed by the s-wave Kaluza-Klein modes propagating along

[1]It has been found that a non-extremal brane origin yields a localized graviton of nonzero mass [6].

[2]Other p-brane origins are possible with the inclusion of a naked singularity [7].

the infinite extra dimension. However, it once appeared to be reasonable to assume that a theory with more than $1+3$ large dimensions did not agree with our direct observations.

It was found that the "warp" factor in certain five-dimensional spaces is such that the effects of gravitational modes propagating in the extra dimension is dampened, yielding four-dimensional gravity at low energies. Embedding the brane world in ten or eleven dimensions brings the possibility of additional "warp" factors. In particular, the effects of modes rotating about compact dimensions could be amplified if the corresponding metric elements are multiplied by a "warp" factor which increases rapidly along the infinite extra dimension. Depending on the geometry of the higher-dimensional origin, rotational Kaluza-Klein modes may actually be as significant as the s-wave Kaluza-Klein modes.[3]

This paper is organized as follows. In section $p-1$ we analyse how the rotational graviton modes modify the four-dimensional gravitational potential for a brane world originating from a Dp-brane, where $p=3,4,5$. In these cases, we find that the rotational modes do not contribute significantly to the modification of the gravitational potential relative to the corrections due to the s-wave Kaluza-Klein modes. It is interesting to note that, in some cases, there are rotational modes that are localized on the brane world. In section 5, we are pleasantly surprised to find that the metric for the near-horizon of the M5/M5 system contains a "warp" factor which amplifies the effect of the rotational graviton modes. In this case, the corrective contributions from the s-wave Kaluza-Klein graviton modes and the rotational Kaluza-Klein graviton modes are of the same magnitude. We offer Conclusions in section 6.

2. D3-brane

The D3-brane metric is

$$ds_{10}^2 = H^{-1/2}(-dt^2 + dx_i^2) + H^{1/2}(dr^2 + r^2 d\Omega_5^2)\,, \tag{2.1}$$

where

$$H = 1 + \frac{R^4}{r^4}\,, \tag{2.2}$$

and $i = 1,2,3$. In the near-horizon limit $r \ll R$, the above metric is that of $AdS_5 \times S^5$. This can be expressed as

$$ds_{10}^2 = (1+k|z|)^{-2}(-dt^2 + dx_i^2 + dz^2) + \frac{1}{k^2} d\Omega_5^2\,, \tag{2.3}$$

where

$$\frac{r}{R} = (1+k|z|)^{-1}\,, \tag{2.4}$$

and $R = k^{-1}$. Note that the absolute value in the above coordinate transformations imposes Z_2 symmetry, which has no known source in ten dimensions.[4]

[3]Throughout this paper, we assume that all compact dimensions are the same size, fixed by the length scale $1/k$ in the "warp" factors.

[4]Recently, it has been found that resolved branes on an Eguchi-Hanson instanton dimensionally reduce to five-dimensional domain walls which localize gravity, without the need of an extra source term[8].

The equation of motion for a graviton fluctuation is that of a minimally-coupled scalar:

$$\partial_M \sqrt{-g} g^{MN} \partial_N \Phi = 0 \,. \tag{2.5}$$

We take $\Phi = P_\ell(S^5)(1+k|z|)^{3/2}\psi(z)M(t,x_i)$, where $\Box_{(4)}M = m^2 M$ and $\Box_{(4)}$ is the laplacian on t, x_i. P_ℓ are the Legendre polynomials of S^5. For graviton modes on the metric (2.3) we obtain

$$-\partial_z^2\psi + \left[\frac{(\ell+3/2)(\ell+5/2)k^2}{(1+k|z|)^2} - 3k\delta(z)\right]\psi = m^2\psi \,, \tag{2.6}$$

where ℓ is an integer parametrizing the rotational dynamics about S^5. For the massless modes, the solution is

$$\psi = (1+k|z|)^{-\ell-3/2} + \frac{\ell}{\ell+4}(1+k|z|)^{\ell+5/2} \,. \tag{2.7}$$

For $\ell = 0$, this wave function is normalizable and there is a corresponding localized graviton. However, for $\ell > 0$ the wave function diverges for large z, and thus there is no permissible massless solution. The massive wave functions are given by

$$\begin{aligned}\psi_m = N_m(1+k|z|)^{1/2}\Bigg[&\left(Y_{1+\ell}\left(\frac{m}{k}\right) - \ell\frac{k}{m}Y_{2+\ell}\left(\frac{m}{k}\right)\right)J_{2+\ell}\left(\frac{m}{k}(1+k|z|)\right) + \\ &+\left(\ell\frac{k}{m}J_{2+\ell}\left(\frac{m}{k}\right) - J_{1+\ell}\left(\frac{m}{k}\right)\right)Y_{2+\ell}\left(\frac{m}{k}(1+k|z|)\right)\Bigg] \,,\end{aligned} \tag{2.8}$$

where

$$N_m = \frac{1}{\sqrt{2k}}\left[\left(Y_{1+\ell}\left(\frac{m}{k}\right) - \ell\frac{k}{m}Y_{2+\ell}\left(\frac{m}{k}\right)\right)^2 + \left(\ell\frac{k}{m}J_{2+\ell}\left(\frac{m}{k}\right) - J_{1+\ell}\left(\frac{m}{k}\right)\right)^2\right]^{-1/2} . \tag{2.9}$$

The newtonian gravitational potential between masses M_1 and M_2 can be estimated by

$$\mathrm{U}(r) \sim \sum_{\ell=0}^{\infty}\left(\frac{G_4 M_1 M_2}{r}\mathrm{e}^{-m_\ell r} + \frac{G_5 M_1 M_2}{r}\int_{m_0^2}^{\infty} d(m^2)|\psi_m(z=0,\ell)|^2 \mathrm{e}^{-mr}\right) , \tag{2.10}$$

where m_ℓ is the mass of a bound state and m_0 is the minimum mass for non-localized states, each for a given ℓ. In the present case, there are no massive bound states or mass gaps due to $\ell > 0$ states, so that $m_0 = m_\ell = 0$. The four and five-dimensional Newton constants are related by $G_4 = kG_5$. This yields

$$\mathrm{U}(r) \sim \frac{G_4 M_1 M_2}{r}\left(\left[1 + \frac{c_0}{(kr)^2} + \cdots\right] + \sum_{\ell=1}^{\infty}\left[\frac{c_\ell}{(kr)^{6+2\ell}} + \cdots\right]\right) , \tag{2.11}$$

where c_0 and c_ℓ are constants of order one. The $\ell = 0$ terms were found in [3, 5]. The terms in the second pair of square brackets are the dominant corrections from the rotational graviton modes ($\ell > 0$). As can be seen, the dominant contribution from the $\ell > 0$ modes is smaller than that of the s-wave Kaluza-Klein modes by the rather small factor $1/(kr)^6$.

3. D4-brane

The metric for the D4-brane is

$$ds_{10}^2 = H^{-3/8}(-dt^2 + dx_i^2) + H^{5/8}(dr^2 + r^2 d\Omega_4^2)\,, \tag{3.1}$$

where

$$H = 1 + \frac{R^3}{r^3}\,, \tag{3.2}$$

and $i = 1, \ldots, 4$. In the near-horizon limit $r \ll R$, the above metric can be expressed as

$$ds_{10}^2 = (1 + k|z|)^{-9/4}(-dt^2 + dx_i^2 + dz^2) + (1 + k|z|)^{-1/4}\frac{1}{k^2}d\Omega_5^2\,, \tag{3.3}$$

where

$$\frac{r}{R} = (1 + k|z|)^{-2}\,, \tag{3.4}$$

The analysis is similar to the case of the D3-brane. An additional element is that the worldvolume direction x_4 is taken to be compact, in order for the worldvolume of the braneworld to have 1+3 dimensions. Rotational dynamics around x_4 are parametrized by the integer n. For simplicity, we will assume that all compact dimensions are of the same size. From (2.10) we find that

$$\mathrm{U}(r) \sim \frac{G_4 M_1 M_2}{r} \sum_{n=0}^{\infty} \mathrm{e}^{-nkr} \left(\left[1 + \frac{b_0}{(kr)^4} + \cdots \right] + \sum_{\ell=1}^{\infty} \left[\frac{b_\ell}{(kr)^{2\sqrt{\ell^2+3\ell+9}+2}} + \cdots \right] \right), \tag{3.5}$$

where b_0 and b_ℓ are constants of order one. The $\ell = 0, M = 0$ terms were found in [5]. The terms in the second pair of square brackets are the dominant corrections for the rotational graviton modes ($\ell > 0$). As can be seen, the dominant contribution from the $\ell > 0$ modes is smaller than that of the s-wave Kaluza-Klein modes by a rather small factor of approximately $1/(kr)^5$. An additional element here, as opposed to the D3-brane origin, is the presence of massive bound states and additional Kaluza-Klein states, both due to the $n > 0$ modes. These contribute Yukawa-like terms to the gravitational potential. In the case of an $M5$-brane, the result is the same as above, except that now there is the possibility of modes rotating around two compact worldvolume dimensions, so that the n in (3.5) is replaced by $\sqrt{n_1^2 + n_2^2}$.

4. D5-brane

The metric for the D5-brane is

$$ds_{10}^2 = H^{-1/4}(-dt^2 + dx_i^2) + H^{3/4}(dr^2 + r^2 d\Omega_3^2)\,, \tag{4.1}$$

where

$$H = 1 + \frac{R^2}{r^2}\,, \tag{4.2}$$

and $i = 1, \ldots, 5$. In the near-horizon limit $r \ll R$, the above metric can be expressed as

$$ds_{10}^2 = \mathrm{e}^{-k|z|/4}\left(-dt^2 + dx_i^2 + dz^2 + \frac{1}{k^2}d\Omega_3^2\right), \tag{4.3}$$

where

$$\frac{r}{R} = \mathrm{e}^{-k|z|/2}, \tag{4.4}$$

In order to yield a braneworld with 1+3 worldvolume dimensions, we take x_4 and x_5 to be compact. We take $\Phi = \mathrm{e}^{i(n_1x_4+n_2x_5)}P_\ell(S^3)\mathrm{e}^{k|z|/2}\psi(z)M(t,x_i)$. For graviton modes on the metric (4.3) we obtain

$$-\partial_z^2\psi + \left[\left(\frac{1}{4}+\alpha^2\right)k^2 - k\delta(z)\right]\psi = m^2\psi, \tag{4.5}$$

where $\alpha^2 \equiv \ell(\ell+2) + n_1^2 + n_2^2$. ℓ, n_1 and n_2 are integers parametrizing the rotational dynamics around S^3, x_4 and x_5 respectively. For the massless modes, the solution is

$$\psi = \mathrm{e}^{-\sqrt{\alpha^2+1/4}k|z|} + A\mathrm{e}^{\sqrt{\alpha^2+1/4}k|z|}, \tag{4.6}$$

where

$$A \equiv \frac{2\sqrt{\alpha^2+1/4}-1}{2\sqrt{\alpha^2+1/4}+1}. \tag{4.7}$$

For $\ell, n_1, n_2 = 0$, this wave function is normalizable and there is a corresponding localized graviton. However, for all higher angular momentum modes, the wave function diverges for large z and thus there is no permissible massless solution. Nonzero ℓ, n_1, n_2 effectively decrease the mass term in the wave equation (4.5). Thus, a higher angular momentum mode has a localized state on the brane world of mass $m^2 = (\ell(\ell+2) + n_1^2 + n_2^2)k^2$. Also, each mode has an equal mass gap between the bound state and a continuum of Kaluza-Klein modes.

The massive wave functions are given by

$$\begin{aligned}
\psi_m &= N_m(k\sin q|z| - 2q\cos q|z|), \\
N_m &= \frac{1}{2\sqrt{\pi q(m^2-\alpha^2k^2)}}, \\
q &= \sqrt{m^2 - k^2\left(\frac{1}{4}+\alpha^2\right)}.
\end{aligned} \tag{4.8}$$

From (2.10) we find that

$$\mathrm{U}(r) \sim \frac{G_4M_1M_2}{r}\sum_{\ell,n_1,n_2=0}^{\infty}\left[\mathrm{e}^{-\alpha kr} + \frac{2}{\sqrt{\pi}}\left(1+4\alpha^2\right)^{3/4}\mathrm{e}^{-\sqrt{\alpha^2+1/4}kr}\left(\frac{1}{(kr)^{3/2}}+\cdots\right)\right], \tag{4.9}$$

The $\ell, n_1, n_2 = 0$ terms were found in [5]. As can be seen, the rotational graviton modes yield Kaluza-Klein potential terms whose dominant correction is smaller than that of the s-wave Kaluza-Klein terms by a relative Yukawa-like factor of approximately e^{-kr}. Notice that there are also contributions from rotational bound states, whose dominant term is smaller than that of the s-wave Kaluza-Klein modes by a relative Yukawa-like factor of $\mathrm{e}^{-kr/2}$.

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5. M5/M5

The metric for two M5-branes intersecting over 1+3 dimensions is

$$ds_{11}^2 = H_1^{-\frac{1}{3}} H_2^{-\frac{1}{3}} \left(-dt^2 + dx_i^2 + H_1(dy_1^2 + dy_2^2) + H_2(dy_3^2 + dy_4^2) + H_1 H_2 (dr^2 + r^2 d\Omega_2^2)\right), \tag{5.1}$$

where

$$H_{1(2)} = 1 + \frac{R_{1(2)}}{r}, \tag{5.2}$$

and $i = 1, 2, 3$. For $R_1 = R_2 \equiv R$ and in the near-horizon limit $r \ll R$, the above metric can be expressed as

$$ds_{11}^2 = \mathrm{e}^{-\frac{2}{3}k|z|} \left(-dt^2 + dx_i^2 + dz^2 + \frac{1}{k^2} d\Omega_2^2\right) + \mathrm{e}^{\frac{1}{3}k|z|} dy_j^2 , \tag{5.3}$$

where

$$\frac{r}{R} = \mathrm{e}^{-k|z|} , \tag{5.4}$$

and $j = 1, \ldots, 4$. We take y_j to be compact. Notice that the metric elements corresponding to the compact M5-brane worldvolume directions y_j are multiplied by a "warp" factor, which increases rapidly with $|z|$. It will be shown how this "warp" factor amplifies the effect that the graviton modes rotating about y_j have on the corrections to the four-dimensional gravitational potential.

We take $\Phi = \mathrm{e}^{i(n_j y_j)} P_\ell(S^2) \mathrm{e}^{k|z|/2} \psi(z) M(t, x_i)$, where we shall implicitly sum over j hereafter. For graviton modes on the metric given by (5.3), we obtain

$$-\partial_z^2 \psi + \left[\left(\ell + \frac{1}{2}\right)^2 k^2 + n_j^2 - k^2 \mathrm{e}^{-k|z|} k \delta(z)\right] \psi = m^2 \psi , \tag{5.5}$$

The solution is

$$\begin{aligned} \psi = N_m \Big[& \left((i\gamma + 1) Y_{i\gamma}(2in) - 2in Y_{i\gamma-1}(2in)\right) J_{i\gamma}\left(2in\mathrm{e}^{-k|z|/2}\right) + \\ & + \left(2in J_{i\gamma-1}(2in) - (i\gamma + 1) J_{i\gamma}(2in)\right) Y_{i\gamma}\left(2in\mathrm{e}^{-k|z|/2}\right)\Big] , \end{aligned} \tag{5.6}$$

where $\gamma \equiv \sqrt{4m^2/k^2 - (2\ell + 1)^2}$ and $n^2 \equiv n_j^2$. The higher ℓ modes contribute massive bound states at $m^2 = \ell(\ell+1)k^2$ and equal mass gaps between the bound states and continua of Kaluza-Klein modes. The $n > 0$ modes only contribute the latter. For $m > (\ell + 2)k$, the Kaluza-Klein modes can be plane-wave normalized with

$$\begin{aligned} N_m = \frac{1}{2\sqrt{k}} \Big[& \sinh(\pi\gamma) \left|(i\gamma + 1) Y_{i\gamma}(2in) - 2in Y_{i\gamma-1}(2in)\right|^2 + \\ & + \frac{1}{\sinh(\pi\gamma)} \left|2in J_{i\gamma-1}(2in) - (i\gamma + 1) J_{i\gamma}(2in)\right|^2 \Big]^{-1/2} . \end{aligned} \tag{5.7}$$

From (2.10) we find that

$$
\begin{aligned}
\mathrm{U}(r) \sim \frac{G_4 M_1 M_2}{r} \sum_{\ell=0}^{\infty} \Bigg[& \mathrm{e}^{-\sqrt{\ell(\ell+1)}kr} + \\
+ \left(\frac{2}{\sqrt{\pi}} (2\ell+1)^{\frac{3}{2}} + \sqrt{2}\pi^{\frac{3}{2}} (2\ell+1) \sum_{j=1}^{4} \sum_{n_j=1}^{\infty} n^2 \left| \frac{J_{-1}(2in)Y_0(2in) - J_0(2in)Y_{-1}(2in)}{2inY_{-1}(2in) - J_0(2in)} \right|^2 \right) & \times \\
\times \mathrm{e}^{-(\ell+1/2)kr} \left(\frac{1}{(kr)^{3/2}} + \cdots \right) \Bigg], & \qquad (5.8)
\end{aligned}
$$

For the D5 and M5/M5 origins, the contributions to the gravitational potential are the same for the $\ell, n = 0$ modes [5] but different for the rotational modes. As can be seen, the $\ell > 0$ modes yield potential terms that are sub-leading to the s-wave terms by a relative Yukawa-like factor, as in the case of the D5-brane origin. Notice there are contributions from massive bound states as well as additional Kaluza-Klein modes.

The $n > 0$ Kaluza-Klein modes contribute to the newtonian potential on the same order in kr as the s-wave Kaluza-Klein modes! In particular, if we consider the $\ell = 0$ modes, we have

$$
\mathrm{U}(r) \sim \frac{G_4 M_1 M_2}{r} \left[1 + \frac{2}{\sqrt{\pi}} \beta \mathrm{e}^{-kr/2} \left(\frac{1}{(kr)^{3/2}} + \cdots \right) \right], \qquad (5.9)
$$

where the factor $\beta = 1.2$ is due to the presence of the $n = 1$ modes. Of course this factor is larger if the length scale of the y_j directions is larger than $1/k$. Modes with $n > 1$ do not make such a significant contribution.

6. Conclusions

We have found that, for brane worlds originating from the near-horizon of a Dp-brane where $p = 3, 4, 5$, the dominant correction of the four-dimensional gravitational potential arises from s-wave Kaluza-Klein modes propagating in the infinite extra dimension. The effect of Kaluza-Klein modes rotating about compact dimensions is sub-leading to that of the s-wave Kaluza-Klein modes, regardless of whether the former states are bound to the brane world (with Yukawa-like factors) or are propagating in the infinite extra dimension. This result is perhaps to be expected.

On the other hand, for a brane world scenario embedded in the near-horizon region of an M5/M5 intersection, we have found that there is a "warp" factor which amplifies the effect of graviton modes rotating about compact directions that lie in the worldvolumes of the M5-branes. In this case, in fact, the modification of the four-dimensional gravitational potential due to these rotational modes is on the same order as that due to the s-wave Kaluza-Klein states. This can be seen explicitly by the presence of β in (5.9). It is rather surprising that, the effects of modes around compact directions can be considerably amplified by "warp" factors which depend on an extra large or infinite dimension. More complicated geometries and compactification schemes could yield additional amplifications in the gravitational

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